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BAYESIAN INFERENCE ON PARAMETERS ASSOCIATED WITH FAMILIES CLOSE--ETC(U)

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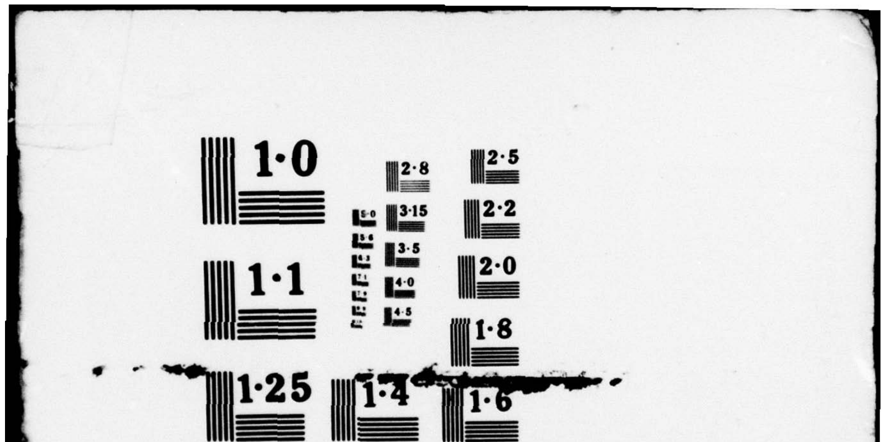
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BAYESIAN INFERENCE ON PARAMETERS ASSOCIATED WITH FAMILIES CLOSED UNDER RECIPROCATION

Norman R. Draper and Irwin Guttman

Mathematics Research Center
University of Wisconsin—Madison
610 Walnut Street
Madison, Wisconsin 53706

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ABSTRACT

Recently, Saunders 1974 has generalized the so-called reciprocal property for normality. Distributions having this property are called ξ-normal, and it is of interest to make statistical inferences about the relevant parameters when sampling from such a distribution. Previous work on such problems has been from the sampling viewpoint.

In this paper, we approach the inference problem from the Bayesian point of view and investigate the posterior of the parameters involved when sampling is from the ξ-normal with parameters α and β. Two special cases are examined, $\xi(v) = \log_e v$, which gives rise to the lognormal distribution, and $\xi(v) = v^{1/2} - v^{-1/2}$, a case that arises in certain fatigue problems (Saunders and Birnbaum 1969).

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SIGNIFICANCE AND EXPLANATION

Statistical problems involving "time to failure", "strength of material", and "fatigue" are often characterized by a statistical distribution known as the lognormal distribution; the length of life of a system subjected to fluctuating stresses by a periodic loading can sometimes be predicted through the use of the Birnbaum-Saunders life length distribution.

These distributions are special cases of a more general class of distributions based on certain functions $\xi(t)$, that map $0 \leq t \leq \infty$ into $-\infty \leq \xi(t) \leq \infty$ and have the property that $\xi(t) = -\xi(1/t)$. In the above cases, we have $\xi(t) = \ln t$ and $\xi(t) = t^{1/2} - t^{-1/2}$, respectively.

A random variable T is said to be ξ-normally distributed if $\xi(T/\beta)/\alpha = Z$, where Z is distributed normally with zero mean and unit standard deviation; α and β are the parameters of the ξ-normal distribution. Given a set of data values $t_1, t_2, \ldots, t_n$, we would like to be able to make inferences about the values of the parameters α and β. In this paper, we approach this problem via Bayesian methods.

The responsibility for the wording and views expressed in this descriptive summary lies with MRC, and not with the authors of this report.

BAYESIAN INFERENCE ON PARAMETERS ASSOCIATED WITH FAMILIES CLOSED UNDER RECIPROCATION*

Norman R. Draper and Irwin Guttman**

1. INTRODUCTION

Recent studies on reliability analysis in fatigue testing have focused on random variables distributed under the ξ-normal distribution; see, for example, Birnbaum and Saunders 1968, 1969; Esary and Marshall 1970; Marshall and Proschan 1972; Saunders and Birnbaum 1969; and Whittaker and Besuner 1969. A precise definition, given by Saunders 1974 may be stated as follows.

Definition 1. Suppose ξ is a concave, monotone increasing, differentiable function from $(0,\infty)$ onto $(-\infty,\infty)$, and such that

$$\xi(t) = -\xi(1/t), \quad t > 0. \tag{1.1}$$

Then the non-negative random variable T is said to have the ξ-normal distribution with parameters (α,β), where $\alpha > 0$, $\beta > 0$, if

$$\frac{1}{\alpha}\,\xi(T/\beta) = Z, \tag{1.2}$$

where Z is a standard normal random variable, that is $Z \sim N(0,1)$.

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An equivalent definition statement is that the non-negative random variable T is ξ-normal if

$$T = \beta\psi(\alpha Z), \quad \alpha,\beta > 0, \tag{1.3}$$

where $\psi = \xi^{-1}$ is a convex monotone increasing transformation from $(-\infty,\infty)$ onto $(0,\infty)$ with the property that

$$\psi(-v) = 1/\psi(v) \tag{1.4}$$

and where $Z \sim N(0,1)$. It is convenient to use (1.3) to establish the first part of the following theorem, due to Birnbaum and Saunders 1969 and Saunders 1974. The second part is obvious.

Theorem 1. If T is ξ-normal with parameters $\alpha,\beta > 0$, then $1/T$ is ξ-normal with parameters α, $1/\beta$. Furthermore, for any real $a > 0$, aT is ξ-normal with parameters $\alpha, a\beta$.

To quote Saunders 1974, p. 534, the above shows that "... the parameter β is, in fact, a scale factor for any ξ-normal distribution." Using (1.1), it is quickly verified that β is "... also the median. It is called the characteristic life in many applications". To see that β is the median, we note from (1.3) and the fact that $\psi = \xi^{-1}$, that

$$P(T \leq \beta) = P(Z \leq \alpha^{-1}\xi(1)). \tag{1.5}$$

However, it is obvious from (1.1) that $\xi(1) = 0$, whereupon the result follows.

In this paper, we discuss the Bayesian approach to inferences about α and β. In Section 2, we obtain the joint posterior distribution of (α,β), and the

marginal distributions of α and β, in general. In Section 3, we examine two commonly occurring special cases:

$$\begin{aligned} &\text{(i)} \quad \xi(v) = \ell n\, v, \\ &\text{(ii)} \quad \xi(v) = v^{1/2} - v^{-1/2}. \end{aligned} \tag{1.6}$$

The former case gives rise to the lognormal distribution, one often assumed in reliability work. The latter case stems from a model for fatigue failure given by Saunders and Birnbaum 1969. It is easily verified that (i) and (ii) satisfy (1.1) and that their inverses satisfy (1.4). Finally, in Section 4, we discuss the question of a Bayesian tolerance interval for β, which has applications to the question of warranty life of a system involving k loading spectra.

2. THE POSTERIOR OF (α,β)

The probability density function (pdf) of Z, the standard normal variable, is

$$\phi(z) = (2\pi)^{-1/2} \exp\{-z^2/2\}. \tag{2.1}$$

The Jacobian of the transformation (1.3) is, from (1.2),

$$\frac{dz}{dt} = \{\xi'(t/\beta)\}/(\alpha\beta), \tag{2.2}$$

where the prime denotes differentiation with respect to t. This derivative is positive since ξ is increasing. Hence the pdf of T, given α,β, is

$$p(t|\alpha,\beta) = \phi[\alpha^{-1}\xi(t/\beta)]\xi'(t/\beta)/(\alpha\beta) \tag{2.3}$$

if $t > 0$, and zero otherwise. Note that, in view of (2.1), (2.3), and (1.3), we may express the mean and variance of the distribution (2.3) as

$$\begin{aligned} E(t|\alpha,\beta) &= \int_0^\infty t\, \phi[\alpha^{-1}\xi(t/\beta)]\xi'(t/\beta)/(\alpha\beta)dt \\ &= \int_{-\infty}^{\infty} \beta\psi(\alpha z)\, \phi(z)dz, \end{aligned} \tag{2.4a}$$

and

$$Var(t|\alpha,\beta) = \int_0^\infty t^2\ \phi[\alpha^{-1}\xi(t/\beta)]\xi'(t/\beta)/(\alpha\beta)dt - [E(t|\alpha,\beta)]^2$$

$$= \int_{-\infty}^\infty \beta^2\psi^2(\alpha z)\phi(z)dz - [E(t|\alpha,\beta)]^2, \tag{2.4b}$$

respectively. That is, (2.4a) and (2.4b) are functionals of ξ (or, equivalently ψ), and can be evaluated for specific ξ (or ψ); see Section 3 for examples.

Suppose now that, prior to taking observations on the random variable T, whose pdf is given by (2.3), very little is known about the parameters α and β. In this situation, we can use the Jeffreys invariant prior

$$p(\alpha,\beta) \propto |I(\alpha,\beta)|^{1/2} \tag{2.5}$$

where I is the usual information matrix consisting of negative expected (with respect to t) second derivatives (with respect to α and or β) of $\ell np(t|\alpha,\beta)$. The likelihood function ℓ of (α,β) for n observations is

$$\ell(\alpha,\beta|\underset{\sim}{t}) \propto (\alpha\beta)^{-n} \prod_{i=1}^{n} \{\phi[\alpha^{-1}\xi(t_i/\beta)]\xi'(t_i/\beta)\}. \tag{2.6}$$

Combining (2.5) with (2.6) via Bayes' Theorem leads to the posterior of (α,β), given by

$$p(\alpha,\beta|\underset{\sim}{t}) \propto (\alpha\beta)^{-n}[\prod_{i=1}^{n} \phi[\alpha^{-1}\xi(t_i/\beta)]\xi'(t_i/\beta)]\ |I(\alpha,\beta)|^{1/2} \tag{2.7}$$

for $\alpha,\beta > 0$, and zero otherwise. In general, the marginal posterior distributions

of α and of β are obtained by integrating out the unwanted parameter. In cases where α and β are independent, a priori, so that

$$|I(\alpha,\beta)|^{1/2} \propto h_1(\alpha)h_2(\beta), \tag{2.8}$$

a situation that occurs in the two examples we provide later, we may express the marginal (posterior) distribution of the median β, the characteristic life, as

$$p(\beta|\underset{\sim}{t}) \propto \beta^{-n}h_2(\beta) \left[\prod_{i=1}^{n} \xi'(t_i/\beta)\right] \int_0^\infty \alpha^{-n}h_1(\alpha) \exp\{-\xi^2(t_i/\beta)/2\alpha^2\}d\alpha, \tag{2.9}$$

and similarily, the marginal (posterior) of α becomes

$$p(\alpha|\underset{\sim}{t}) \propto \alpha^{-n}h_1(\alpha) \int_0^\infty \beta^{-n}h_2(\beta) \left[\prod_{i=1}^{n} \{\phi[\alpha^{-1}\xi(t_i/\beta)]\xi'(t_i/\beta)\}d\beta.\right. \tag{2.10}$$

It is interesting to note from (2.7) that the parameter β has a marginal (posterior) distribution that exhibits a reciprocal property. For if we set $\beta = \theta^{-1}$, with Jacobian $\partial\beta/\partial\theta = -\theta^{-2}$, and use the facts that

$$\begin{aligned} \theta^2 \, |I(\alpha,\theta)|^{1/2} &= |I(\alpha,\beta)|^{1/2}, \\ \phi(z) &= \phi(-z), \\ \xi(t/\beta) &= -\xi(\beta/t), \\ t^2\xi'(t/\beta) &= \beta^2\xi'(\beta/t), \end{aligned} \tag{2.11}$$

we obtain a joint distribution $p(\alpha,\theta|\underset{\sim}{r})$ with $\underset{\sim}{r} = (r_1,r_2,\ldots,r_n)'$ and $r_i = 1/t_i$ of exactly the same form as (2.7). In other words the posterior distributions of β and of β^{-1} belong to the same family, a property that arises from the assumption of ξ-normality.

Tolerance regions for future life-lengths.

We here review standard definitions; see Guttman (1967).

Definition 2: Suppose $(t_1,t_2,\ldots,t_n) = \underset{\sim}{t}'$ is a vector of n independent observations sampled from the distribution $p(t|\underset{\sim}{\theta})$. Then a region of t values defined by a function $S(t_1,t_2,\ldots,t_n) \subset R_1 \equiv \{t|-\infty < t < \infty\}$ is said to have coverage C if

$$C = C(S) = \int_S p(t|\underset{\sim}{\theta})dt. \tag{2.12}$$

Note that C, a functional of S, is a function of $\underset{\sim}{\theta}$.

Definition 3: S is a δ-expectation tolerance region if

$$E\{C(S)|t\} = \delta \tag{2.13}$$

the expectation being taken with respect to the posterior distribution $p(\underset{\sim}{\theta}|\underset{\sim}{t})$.

A procedure for the construction of an S satisfying (2.13) is as follows. (For proofs and discussion, see Guttman 1967; 1970, pp. 132-3.)

1. Find the predictive distribution

$$h(t|\underset{\sim}{t}) = \int_{\underset{\sim}{\theta}\epsilon\Omega} p(t|\underset{\sim}{\theta})p(\underset{\sim}{\theta}|\underset{\sim}{t})d\underset{\sim}{\theta} = \int_{\underset{\sim}{\theta}\epsilon\Omega} p(\underset{\sim}{\theta},t|\underset{\sim}{t})d\underset{\sim}{\theta} \tag{2.14}$$

where Ω denotes the parameter space.

2. Find S, a δ-confidence region for t such that

$$P(t\epsilon S|\underset{\sim}{t}) = \int_S h(t|\underset{\sim}{t})dt = \delta \ . \tag{2.15}$$

For our problem, with $\underset{\sim}{\theta} = (\alpha,\beta)'$, $p(t|\underset{\sim}{\theta})$ is (2.3) and $p(\underset{\sim}{\theta}|\underset{\sim}{t})$ is (2.7) assuming the prior of (2.5). It follows that $p(\underset{\sim}{\theta},\underset{\sim}{t}|t)$ is of the form of (2.7) with $(n+1)$ replacing n and the vector $\underset{\sim}{t}$ extended to $(t_1,t_2,\ldots,t_n,t) = \underset{\sim}{t}'_{n+1}$, say. Equation (2.14) then defines the normalizing constant of this distribution. This process will be illustrated in Section 3.

3. TWO SPECIAL CASES

The Log-Normal Distribution.

In many reliability papers, the distribution of the random variable being studied (e.g., time to failure, strength, fatigue) is assumed to be log-normal. That is,

$$\xi(v) = \ell n v \tag{3.1}$$

so that, in the notation of Definition 1,

$$\alpha^{-1} \ell n(T/\beta) = Z, \quad \text{or} \quad T = \beta \exp\{\alpha Z\}, \tag{3.2}$$

where $Z \sim N(0,1)$. If we apply the foregoing work, we find a number of well-known results. The invariant prior is proportional to $\alpha^{-2}\beta^{-1}$ and, a posteriori, $W/\alpha^2 \sim \chi_n^2$ where $W = \Sigma(u_i-\bar{u})^2$ and $u_i = \ell n t_i$, and $n(\ell n\beta-\bar{u})/W \sim t_n$. We omit the details; for closely related results with a different prior, and for a number of other developments, see Zellner (1971). For comments on the improper prior distribution, see the second example.

The Birnbaum-Saunders Life Length Distribution.

Birnbaum and Saunders 1969 are concerned with a random variable T, the length of life of a system which is subjected to fluctuating stresses by a periodic loading. Under various assumptions, they show that, in the language of Definition 1, T is ξ-normal with

$$\xi(v) = v^{1/2} - v^{-1/2}, \tag{3.3}$$

that is, T is such that

$$\alpha^{-1}\xi(T/\beta) = \alpha^{-1}[(T/\beta)^{1/2} - (\beta/T)^{-1/2}] = Z, \tag{3.4}$$

It is straightforward, but tedious, to verify that, for (3.4)

$$T = \begin{cases} \frac{1}{2}\beta[(\alpha Z)^2+2 \ - \{(\alpha Z)^4+4(\alpha Z)^2\}^{1/2}] & \text{if } Z \leq 0, \\ \\ \frac{1}{2}\beta[(\alpha Z)^2+2 \ + \{(\alpha Z)^4+4(\alpha Z)^2\}^{1/2} & \text{if } Z > 0, \end{cases} \tag{3.5}$$

so that (see (1.3) with $W = \alpha Z$),

$$\psi(W) = \begin{cases} \frac{1}{2}[W^2+2 \ - \{W^4+4W^2\}^{1/2}] & \text{if } W \leq 0, \\ \\ \frac{1}{2}[W^2+2 \ + \{W^4+4W^2\}^{1/2}] & \text{if } W > 0, \end{cases} \tag{3.6}$$

is convex, monotone increasing, and such that

$$\psi(W) = 1/\psi(-W). \tag{3.7}$$

Using properties of $Z = N(0,1)$ we find, from (3.5), that

$$E(T|\alpha,\beta) = \beta\ \{1+\tfrac{1}{2}\alpha^2\},$$

$$E(T^2|\alpha,\beta) = \beta^2\{1+2\alpha^2+3\alpha^4/2\}, \tag{3.8}$$

so that

$$\mathrm{Var}(T|\alpha,\beta) = \beta^2\alpha^2\{1+5\alpha^2/4\}, \tag{3.9}$$

with

$$\{\mathrm{Var}(T|\alpha,\beta)\}^{1/2}/E(T|\alpha,\beta) = \alpha(1+5\alpha^2/4)^{1/2}/(1+\alpha^2/2), \tag{3.10}$$

a function of α alone. The results (3.8), (3.9) were found by Birnbaum and Saunders 1969, using a different method. Of course, if we substitute (3.4) in (2.3), we find immediately that

$$p(t|\alpha,\beta) = \{(t+\beta)/[\alpha^2\beta\pi(2t)^3]^{1/2}\}\ \exp\{-[(t/\beta)^{1/2}-(\beta/t)^{1/2}]^2/(2\alpha^2)\} \tag{3.11}$$

for $t > 0$, because

$$\xi'(t/\beta) = \beta^{1/2}(t+\beta)/\{2t^{3/2}\} \tag{3.12}$$

and we may find moments directly from (3.5).

The Jeffrey's invariant prior for this problem has the interesting form (details are given in the Appendix):

$$p(\alpha,\beta) \propto |I(\alpha,\beta)|^{1/2} = [\alpha^2(1-G(\alpha))+2]^{1/2}/(\alpha^2\beta) \tag{3.13}$$

where

$$G(\alpha) = \{1-\Phi(2/\alpha)\}/\{\alpha\phi(2/\alpha)\}, \tag{3.14}$$

with $\Phi(u) = \int_{-\infty}^{u} \phi(z)dz$. If we set $x = 2/\alpha$, (3.14) takes the form

$$G(2/x) = \tfrac{1}{2}x\{1-\Phi(x)\}/\phi(x) = \tfrac{1}{2}xR(x) \tag{3.15}$$

where $R(x)$ is <u>Mill's ratio</u> as given by Johnson and Kotz 1970, p. 278, (72).

The prior (3.13) is improper, that is, the integral with respect to α and β diverges; this is true with respect to the parameters individually, in fact. In most practical cases, this is not a matter for concern. Improper priors

often arise in the expression of imprecise prior knowledge, and correspond to "flat" prior information in some suitably transformed parameter space. Once the information in the likelihood is factored in, the consequent posterior typically converges. If it did not, the sample information would be extremely weak and of the same character as the prior information. Improper priors can be regarded as approximations to proper priors in situations where the proper prior is "flat" in the region covered by the likelihood. In situations where a Bayesian solution is paralleled by the sampling theory solution, improper priors frequently occur in this way. In such cases, criticism of an improper prior solution implies criticism of the sampling theory solution as well. For recent discussion on the various aspects of improper priors, see Dawid, Stone and Zidek (1973).

Suppose now that n independent observations $(t_1, t_2, \ldots, t_n) = \underset{\sim}{t}'$ are taken on T, whose distribution is (3.11). From (2.7), (3.12) and (3.13), and after some simple algebra, the posterior of (α, β), given $\underset{\sim}{t}$, is seen to be

$$p(\alpha,\beta|\underset{\sim}{t}) = K\alpha^{-(n+2)}\beta^{-(n+2)/2}\{\alpha^2[1-G(\alpha)]+2\}^{1/2}[\prod_{i=1}^{n}(\beta+t_i)]$$
$$\times \exp\{-[S_2(\beta-\beta_0)^2+V]/[2\alpha^2\beta]\}, \tag{3.16}$$

where K is a normalizing constant,

$$V = S_1 - n^2/S_2 = (S_1S_2-n^2)/S_2, \tag{3.17}$$

$$S_1 = \sum_{i=1}^{n} t_i, \quad S_2 = \sum_{i=1}^{n} t_i^{-1}, \text{ and } \beta_0 = n/S_2. \tag{3.18}$$

Using the Schwarz inequality, it is easily shown that, because $t_i \geq 0$, $V \geq 0$.

For this case, the marginal distributions $p(\alpha|\underset{\sim}{t})$ and $p(\beta|\underset{\sim}{t})$ are not

convenient explicit forms and so, in working our illustrative example below, we have integrated (3.16) numerically over β and α respectively. (We could, of course, express $p(\alpha|t)$ in series form explicitly.)

Numerical Illustration

We used the Rand Corporation Tables 1955 to generate 15 random z_i's which were converted into t_i's via (3.5). These are given in Table 3.1; it follows that $n = 15$, $V = 454.39$, $S_1 = 498.50$ $S_2 = 5.10$, and $\beta_o = 2.94$, to two decimal places.

Table 1. Fifteen independent observations of life length.

45.3	35.8	115.7	99.9	52.5	66.9	42.4	13.3
7.1	0.4	1.8	8.5	1.0	2.5	5.4	

The posterior distributions for α and β, obtained by numerical integration of (3.16) with respect to β and α, respectively, are shown in Figures 1(a) and (b). From this calculation, we obtain 95% posterior confidence intervals for α and β as

$$1.73 \leq \alpha \leq 3.38 \qquad 4.57 \leq \beta \leq 16.59$$

the true values being 2 and 10 respectively.

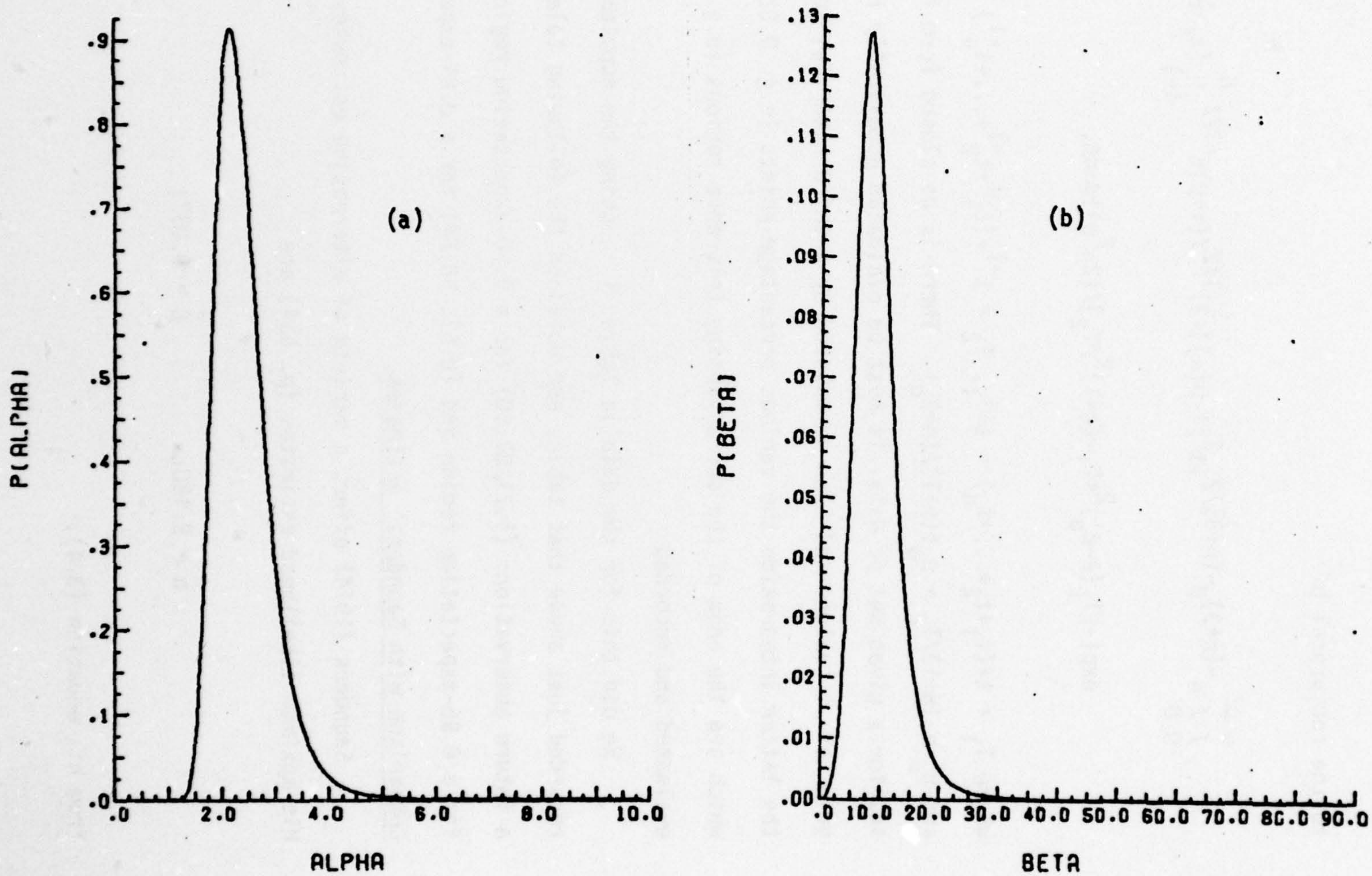
The mean and mode of $p(\alpha|\underset{\sim}{t})$ are at 2.41 and about 2.17, while the mean and mode of $p(\beta|\underset{\sim}{t})$ are at 9.53 and about 8.02, respectively.

The question of suitable approximations for the marginals remains to be answered.

Tolerance regions.

For this example, $h(t|\underset{\sim}{t})$ of (2.14) is, by extension of (3.43), proportional

Figure 1. Posterior distributions of α and β for the life length numerical example: (a) $p(\alpha|\underset{\sim}{t})$, (b) $p(\beta|\underset{\sim}{t})$.



to the reciprocal of

$$\int_0^\infty \int_0^\infty \alpha^{-(n+3)}\beta^{-(n+3)/2}\{\alpha^2[1-G(\alpha)]+2\}^{1/2}/(t+\beta)t^{-3/2} \prod_{i=1}^{n} (t_i+\beta)t_i^{-3/2} \times$$

$$\exp\{-[T_2(\beta-\beta_0')^2+T_1-(n+1)^2/T_2]/(2\alpha^2\beta)\}d\alpha d\beta, \tag{3.19}$$

where $T_1 = t+(t_1+t_2+\ldots+t_n) = t+S_1$, $T_2 = t^{-1}+(t_1^{-1}+t_2^{-1}+\ldots+t_n^{-1}) = t^{-1} + S_2$ and $\beta_0' = (n+1)/T_2 = \beta_0 t(n+1)/(n+\beta_0)$. There is no closed form for (3.19) and so, for a given set of data, it must be evaluated numerically for a selected grid of t values and then normalized by a third numerical integration. During the latter integration the various percentage points (e.g. 0.025 and 0.975) which are the ends of the corresponding tolerance regions (e.g. 95%) can also be evaluated and recorded.

We did this for the data in Table 1 using the supplementary figures recorded just above that table, and obtained the following tolerance regions for a future observation: (1.71, 87.50) for a 0.90-expectation region, (1.10, 93.47) for a 0.95-expectation region and (0.52, 98.64) for a 0.99-expectation region.

Comparison with Saunders' estimates.

Saunders (1974) offers a variety of alternative estimates for β and α. His maximum likelihood estimates (p. 534) are

$$\hat{\alpha} = 2.180, \qquad \hat{\beta} = 9.037;$$

from his equation (3.4),

$$\tilde{\beta} = 9.886;$$

from the third display on p. 535

$$\alpha^* = 5.855, \qquad \beta^* = 11.343;$$

from p. 535 in Remark 3.6,

$$\breve{\beta} = 12.280;$$

and from p. 536, second display

$$\beta^0 = 13.3, \text{ the middle } T_i.$$

We see that only $\hat{\alpha}$, $\hat{\beta}$, and $\tilde{\beta}$ provide values that appear appealing compared with those we obtain from our Bayesian approach, and from our knowledge of the true values $\alpha = 2$, $\beta = 10$.

APPENDIX

Jeffrey's invariant prior $p(\alpha,\beta)$ for Section 3, second example.

From (3.11) we have,

$$L = \ell np = \text{constant} + \ell n(t+\beta) - \ell n\alpha - \tfrac{1}{2}\ell n\beta - (t\beta^{-1}+t^{-1}\beta-2)/(2\alpha^2). \qquad \text{(A.1)}$$

The expected (with respect to t) matrix of negative second derivatives is

$$E\begin{bmatrix} -\dfrac{\partial^2 L}{\partial\alpha^2} & -\dfrac{\partial^2 L}{\partial\alpha\partial\beta} \\ -\dfrac{\partial^2 L}{\partial\alpha\partial\beta} & -\dfrac{\partial^2 L}{\partial\beta^2} \end{bmatrix} = E\begin{bmatrix} 3\alpha^{-4}(t\beta^{-1}+t^{-1}\beta-2) - \alpha^{-2} & \alpha^{-3}(t\beta^{-2}-t^{-1}) \\ \alpha^{-3}(t\beta^{-2}-t^{-1}) & (t+\beta)^{-2}+t\alpha^{-2}\beta^{-3}-\tfrac{1}{2}\beta^{-2} \end{bmatrix}$$

$$= \begin{bmatrix} I_{11} & I_{12} \\ I_{12} & I_{22} \end{bmatrix}, \qquad \text{(A.2)}$$

say. Now

$$E(\partial(\ell np)/\partial\alpha) = 0 \text{ implies } E(t\beta^{-1}+t^{-1}\beta-2) = \alpha^2. \qquad \text{(A.3)}$$

It immediately follows that $I_{11} = 2/\alpha^2$. Also

$$E(\partial(\ell np)/\partial\beta) = 0 \quad \text{implies } E(t+\beta)^{-1} - \tfrac{1}{2}E(t^{-1}-t\beta^{-2})/\alpha^2 = \tfrac{1}{2}\beta^{-1} \tag{A.4}$$

and, after some algebra, it can also be shown that

$$E(t+\beta)^{-1} = \tfrac{1}{2}\beta^{-1}. \tag{A.5}$$

It follows from (A.2), (A.4) and (A.5) that $I_{12} = 0$. Now we can show directly that

$$E(t) = \tfrac{1}{2}\beta(2+\alpha^2), \tag{A.5}$$

whereupon, from (A.2),

$$I_{22} = E(t+\beta)^{-2} + \alpha^{-2}\beta^{-2}. \tag{A.6}$$

We now need $E(t+\beta)^{-2}$. This is a tedious calculation in which account must be taken of (3.5), and a series of reductions performed. Eventually we find that

$$E(t+\beta)^{-2} = \tfrac{1}{2}\beta^{-2}\{1-G(\alpha)\} \tag{A.7}$$

where $G(\alpha)$ is defined by (3.14). Combining (A.2), (A.6) and (A.7) and the facts that $I_{11} = 2\alpha^{-2}$, $I_{12} = 0$ now provides the result (3.13).

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19. KEY WORDS (Continue on reverse side if necessary and identify by block number)

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20. ABSTRACT (Continue on reverse side if necessary and identify by block number)

Recently, Saunders 1974 has generalized the so-called reciprocal property for normality. Distributions having this property are called ξ-normal, and it is of interest to make statistical inferences about the relevant parameters when sampling from such a distribution. Previous work on such problems has been from the sampling viewpoint.

In this paper, ~~we~~ approach the inference problem from the Bayesian point of view and investigate the posterior of the parameters involved when sampling is → next page

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20. ABSTRACT - Cont'd.

xi alpha beta

from the ξ-normal with parameters α and β. Two special cases are examined. $\xi(v) = \log_e v$, which gives rise to the lognormal distribution, and $\xi(v) = v^{1/2} - v^{-1/2}$, a case that arises in certain fatigue problems (Saunders and Birnbaum 1969).